DIAS-STP 89-33

On the Functional Integrals Associated to a Special Gibbs Systems with Three Body Potentials.

W.I.Skrypnik

Abstract:

The Lagrangian Euclidean Quantum Field Theory of two interacting vector fields is found, which is equivalent to a special Gibbs system with three body potential.

Institute of Mathematics of the Ukrainian SSR, Repin Street 3, Kiev 4, 252601, USSR.

1. Introduction.

The Sine-Gordon representation for the grand canonical partition function and correlation functions of an r-component Gibbs system with the potential energy

$$U_o((x,\sigma)_n) = \sum_{k<j=1}^{n} \sigma_k \sigma_j c(x_k - x_j), \quad \sigma_j \in \mathbb{R}, \; x_j \in \mathbb{R}^d$$

plays an important role in modern statistical mechanics. With its help the rigorous results for charged systems were obtained [1-6]. The analog of the Sine-Gordon representation for the Gibbs systems with many-body potentials has not been found yet. In this paper we derive this analog for Gibbs systems with two types of three body potentials

$$U((x,\sigma)_n) = U_o((x,\sigma)_n) + \frac{1}{4}\beta \sum_{j=1}^{n} \left\{ \sum_{k=1,\neq j}^{n} \sigma_i \sigma_j c(x_i - x_j) \right\}^2 \qquad (1.1\ \text{i})$$

$$U((x,\sigma)_n) = -\frac{1}{2} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} U_o((x,\sigma)_n) +$$

$$+ \frac{\beta}{4} \sum_{j=1}^{n} \left\{ \frac{\partial}{\partial x_j} U_o((x,\sigma)_n) \right\}^2 , \qquad (1.1\ \text{ii})$$

where

$$(x,\sigma)_n = (x_1\sigma_1; \ldots; x_n\sigma_n),$$

$$U_o((x,\sigma)_n) = \sum_{i<j=1}^{n} \sigma_i \sigma_j c(x_i - x_j)$$

$$\left\{ \frac{\partial}{\partial x} U \right\}^2 = \sum_{\alpha=1}^{d} \left\{ \frac{\partial}{\partial x^\alpha} U \right\} \left\{ \frac{\partial}{\partial x^\alpha} U \right\} ,$$

σ_j is the charge of the j-th particle, and the coefficient $\frac{\beta}{4}$ is introduced for later convenience.

The second expression for the potential energy appears in the heat equation, to which the Smoluchowski equation is reduced by the substitution

$$\Psi(X_n) = \exp \{ - \frac{\beta}{2} U(X_n) \} \quad \tilde{\Psi} (X_n) ;$$

the Smoluchowski equation is the forward Kolmogorov equation for the stochastic (Ghihman-Ito) equation

$$\frac{d}{dt} x_{j,\sigma_j}(t) = - \frac{\partial}{\partial x_{j,\sigma_j}} U((x,\sigma)_n) + \beta^{-\frac{1}{2}} \dot{W}_j (t)$$

where $\{ \dot{W}_j(t) \}$ is the sequence of the independent processes of white noise.

The discussed systems are not difficult to treat in the case of an integrable smooth potential $c(x)$, since they can be reduced to the Gibbs systems with a complex pair potential with the help of the transformation

$$\exp\left\{-\frac{1}{4}\beta^2\sum_{j=1}^{n}\left\{\sum_{j\neq k=1}^{n}\sigma_j\sigma_k c(x_j-x_k)\right\}\right\}=$$

$$=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^d}\exp\left\{i\frac{\beta}{4}\sum_{j=1}^{n}\sum_{j\neq k=1}^{n}\sigma_j\sigma_k(q_j+q_k)\,c(x_j-x_k)\right\}\times$$

$$\times\exp\left\{-\frac{1}{2}\|Q_n\|^2\right\}dQ_n\,, \qquad (1.2\,i)$$

$$\exp\left\{-\frac{\beta^2}{4}\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}U_o((x,\sigma)_n)\right)^2\right\}=$$

$$=(2\pi)^{-\frac{dn}{2}}\int_{\mathbb{R}^{dn}}\exp\left\{i\frac{\beta}{2}\sum_{j=1}^{n}\left(q_j,\frac{\partial}{\partial x_j}U_o((x,\sigma)_n)\right)\right\}\times$$

$$\times\exp\left\{-\frac{1}{2}\|Q_n\|^2\right\}dQ_n\,, \qquad (1.2ii)$$

where $\|Q_n\|^2=\sum_{j=1}^{n}q_j^2$

If $c(x)$ is an integrable function the thermodynamic limit of the correlation function can be found as the solution of the generalized Kirkwood-Saltsbourg equation; these equations do not help much when $c(x)$ is not an integrable function, and in this case we cannot pass to the thermodynamic limit. The proposed Sine-Gordon type representation shows a way of dealing with this limit not only in the case of the charged equilibrium system with the potential energy (1.1) but also in the case of the non-equilibrium diffusion system, mentioned above, when the initial distribution is Gibbsian. It establishes the correspondence between the Gibbs systems (1.1) and the Quantum Euclidean system of two interacting scalar fields $\varphi(x)$, $\varphi_*(x)$ in the case (i) and two vector fields $\varphi(x) = \{ \varphi_1, \ldots \varphi_d \}$, $\varphi_*(x) = \{ \varphi_{*1}, \ldots, \varphi_{*d} \}$ in the case (ii) with Lagrangians, respectively

$$L(\varphi , \varphi) = (C^{-1}\varphi_* , \varphi) +$$

$$+ \sum_{j=1}^{r} \hat{z}_j \int_{\mathbb{R}^d} \exp \left\{ i \frac{\sqrt{\beta}}{2} \sigma_j \phi (x,\varphi) - \frac{\sigma_j^2}{4} \beta \| \varphi_*(x) + \sigma_j C_o \|^2 \right\} dx \tag{1.3}$$

with

(i) $\quad \phi(x,\varphi) = \varphi(x), \quad C_o = c(0) , \quad \hat{z}_j = \exp \left\{ \frac{\beta}{2} \sigma_j^2 c(0) \right\} z_j$

resp.

(ii) $\phi(x,\varphi) = \mathbf{div}\, \varphi(x), \quad C_o = \nabla c(0), \quad \hat{z}_j = \exp\left\{ \frac{\beta}{2} \sigma_j^2 (-\Delta c)(0) \right\} z_j$,

where $(.,.)$ is the scalar product in $L^2(\mathbb{R}^d)$ or $L^2(\mathbb{R}^d)\otimes\mathbb{R}^d$, C^{-1} is the inverse of the operator

$$(Ch)_s(x) = \int_{\mathbb{R}^d} c(x-x') h(x') dx',$$

and $c(x)$ is a positive definite smooth function

The most remarkable and unpleasant feature of the Lagrangians is that they are degenerate in φ, i.e. there are no quadratic terms in φ in them.

2. The main equations. Grand partition function.

To derive the introduced Lagrangians we start from the following identities

$$\sum_{j<k=1}^{n} c(x_j-x_k)\ \sigma_j\sigma_k + \frac{i}{2}\sum_{j<k=1}^{n} \sigma_j\sigma_k(q_j+q_k)\ c(x_j-x_k) =$$

$$= \frac{1}{2}\sum_{j\neq k=1}^{n} \sigma_j\sigma_k(1+iq_j)(1+iq_k)\ c(x_j-x_k) + \frac{1}{2}\sum_{j\neq k=1}^{n} \sigma_j\sigma_k q_j q_k c(x_j-x_k) -$$

$$- \frac{1}{2}\sum_{j=1}^{n}\left(i\ \sigma_j^2\ q_j\ c(0) + \sigma_j^2\ c(0)\right) \qquad . \tag{2.1. i}$$

$$\sum_{j=1}^{n} (\nabla_j U_o((x,\sigma)_n), q_j) + \sum_{k\angle j=1}^{n} \sigma_j \sigma_k (-\Delta c)(x_j - x_k) =$$

$$= \frac{1}{2} \sum_{k\neq j=1}^{n} \sigma_j \sigma_k (\nabla_j + iq_j)(\nabla_k + iq_k) c(x_j - x_k) +$$

$$+ \sum_{k\neq j=1}^{n} \sigma_j \sigma_k (q_j, q_k) c(x_j - x_k) - \frac{1}{2} \sum_{j=1}^{n} \sigma_j^2 \{ (q_j, (\nabla c)(o)) + (-\Delta c)(o) \} \tag{2.1. ii}$$

Let us denote

$$(x, \sigma_j)_n = (x_1 \sigma_{j_1}; \ldots; x_n \sigma_{j_n}).$$

From (1.1-2) and (2.1) it follows that

$$\exp\left\{ - \beta U(x, \sigma_j)_n \right\} =$$

$$= \int \exp\left\{ - \frac{1}{2} \|Q_n\|^2 \right\} dQ_n \int \mu(d\varphi) \int \mu(d\varphi_*) \times$$

$$\times \prod_{l=1}^{n} \exp\left\{ \frac{i}{2} \sqrt{\beta} \sigma_{j_1} \left[\phi(x_1, \varphi), + i ((q_1, \varphi(x_1) + \varphi(x_1)) + \right.\right.$$

$$\left.\left. + \sigma_{j_1} [(q_1, C_o) + i\hat{C}_o] \right] \right\} \tag{2.2}$$

Here $\int$ stands for $(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^d}$, or $(2\pi)^{-\frac{dn}{2}} \int_{\mathbb{R}^{dn}}$, respectively, in the case (i) $(q,C\ )$ denotes qC_0, and we put

$$\text{(i)}\ \hat{C}_0 = C_0 = c(0), \qquad \text{(ii)}\ \hat{C}_0 = (-\Delta c)(0) \quad .$$

Functional measures $\mu(d\varphi)$, $\mu(d\varphi_*)$ define two independent scalar random gaussian fields with the covariance $c(.)$ in the first case and two component-wise independent vector gaussian fields with the same covariance in the second.

Integrating over Q_n variables we obtain

$$\exp\left\{ -\beta U(x\ ,\sigma_j)_n \right\} = \int \mu(d\varphi) \int \mu(d\varphi_*) \prod_{l=1}^{n} \exp\left\{ \frac{i}{2} \sqrt{\beta}\, \sigma_{j_l} \Phi(x_l,\varphi) \right\}$$

$$\times \exp\left\{ -\frac{\beta}{4} \sigma_{j_l} \|\varphi_*(x_l\ ) + \varphi(x_l\ ) + \sigma_{j_l}^2 C_o \|^2 \right\} \exp\left\{ \frac{\beta}{2} \sigma_{j_l}^2 \hat{C}_o \right\}$$

Now let us consider the grand partition function Ξ_Λ

$$\Xi_\Lambda = \sum_{n\geq 0} \sum_{\Sigma n_l = n} \prod_{l=1}^{r} \frac{z^{n_l}}{n_l!} \sum_{(\sigma')_n} \int_{\Lambda^n} \exp\left\{ -\beta U((x^{(r)}\sigma')_n) \right\} \prod_{s=1}^{r} dX_{n_s}^{(s)} \quad ,$$

where $(x^{(r)},\sigma)_n = (x_1^{(1)},\sigma_1;\ldots x_{n_1}^{(1)},\sigma_1),\ldots(x_1^{(r)},\sigma_r, \ldots x_{n_r}^{(r)},\sigma_r)$

With the help of (2.3) we derive the Sine-Gordon representation for Ξ_Λ

$$\Xi_\Lambda = \int \mu(d\varphi) \int \mu(d\varphi_*) \exp\left\{ L'_\Lambda(\varphi,\varphi_*) \right\} \qquad (2.4)$$

$$L'_\Lambda(\varphi,\varphi_*) = \int_\Lambda L'(\varphi(x),\varphi_*(x))dx \ ,$$

$$L(\varphi(x),\varphi_*(x)) = \sum_{j=1}^{r} \hat{z}_j \exp\left\{ \frac{i}{2}\sqrt{\beta}\ \sigma_j \phi(x,\varphi) - \frac{\beta}{4}\sigma_j^2 \| \varphi_*(x)+i\varphi(x)+|\sigma_j| C_o \|^2 \right\}$$

(1.3) follows from (2.4) if we change the variables in the functional integral, making the complex translation [5-6].

3. Correlation functions.

The correlation functions are defined by the following expression

$$\rho_\Lambda((x,\sigma_j)_m) = \Xi_\Lambda^{-1} \left(\prod_{l=1}^{m} z_{j_l} \chi_\Lambda(x_j) \right) \sum_{n\geq 0} \sum_{\Sigma n_k = n} \prod_{s=1}^{r} z_{j_s} (n_s\ !)^{-1} \times$$

$$\sum_{(\sigma')_n} \int_{\Lambda^n} \exp\left\{ -\beta U((x,\sigma_j)_m, (x^{(r)},\sigma')_n) \right\} dX_n$$

where $\chi_\Lambda(x)$ is the characteristic function of compact set Λ, and $X_n = (X^{(1)}_{n_r},\ldots,X^{(r)}_{n_r})$. (2.3) yields the representation

$$\rho_\Lambda((x,\sigma_j)_m) = \Xi_\Lambda^{-1} \int \mu(d\varphi) \int \mu(d\varphi_*) \quad \exp\left\{ L_\Lambda(\varphi,\varphi_*) \right\} \times \tag{3.1}$$

$$\prod_{l=1}^{m} \hat{z}_{j_l} \chi_\Lambda(x_l) \exp\left\{ \frac{i}{2}\sqrt{\beta}\ {}_{j_l}\Phi(x_l,\varphi) - \frac{\beta}{4}\sigma_{j_l} \| \varphi_*(x_l)+i\varphi(x_l)+|\sigma_{j_l}| \hat{C}_o \|^2 \right\}$$

Now let $\eta_\Lambda(x) \in C_o^\infty(\Lambda')$, $\Lambda \subset \Lambda'$; $\eta_\Lambda(x) = 1$, if $x \in \Lambda$. It is clear that nothing changes if we multiply φ by η_Λ. After this let us make a complex translation [2,3]

$$\varphi_*(x) \Rightarrow \varphi_*(x) + i\,\eta_\Lambda(x)\,\varphi(x)$$

As the result we obtain

$$\Xi_\Lambda = \int \mu(d\varphi)\ \exp\left\{\frac{1}{2}\,(C^{-1}\eta_\Lambda\varphi,\ \eta_\Lambda\varphi)\right\} \int \mu(d\varphi_*) \exp\left\{-i(C^{-1}\varphi_*, \eta_\Lambda\varphi)\right\}\times$$

$$\times\exp\left\{L_\Lambda^o(\varphi, \varphi_*)\right\},\quad \rho_\Lambda((x,\sigma_j)_m) = \Xi_\Lambda^{-1}\int \mu(d\varphi)\ \exp\left\{\frac{1}{2}(C^{-1}\eta_\Lambda\varphi,\varphi)\right\}\int \mu(d\varphi_*)\times$$

$$\times\exp\left\{-i(C^{-1}\varphi_*, \eta_\Lambda) + L_\Lambda^o(\varphi,\varphi_*)\right\} \rho_o(\varphi,\varphi_*;(x,\sigma_j)_m), \qquad (3.3)$$

where ρ_o is defined by the previous expression for the correlation functions. Formally

$$\mu(d\varphi)\ \exp\left\{\frac{1}{2}\,(C^{-1}\eta_\Lambda\varphi,\varphi_\Lambda)\right\} \underset{\Lambda\Rightarrow\mathbb{R}^d}{\Rightarrow} \prod_{x\in\mathbb{R}^d} d\varphi(x)$$

So we derived the Lagrangian from the introduction.

For neutral systems we have

$$L_o(\varphi(x),\varphi_*(x)) =$$

$$\sum_{j=1}^{k} \hat{z}_j \cos\left\{ \frac{1}{2} \sqrt{\beta}\, \sigma_j\, \phi(x,\varphi) \right\} \exp\left\{ - \frac{\beta}{4} \sigma_j^2 \|\varphi_*(x) + |\sigma_j| \hat{C}_o \right\}, \quad r=2k.$$

In spite of the fact that the introduced Lagrangians are degenerate the rigorous approach can be developed. It demands that we integrate out first the field φ_* to find the effective Lagrangian $L_\Lambda(\varphi)$. It can be easily found as a bounded functionin the case of integrable potential.

It is worth remarking that the field **div** $\varphi(x)$ has a short range covariance when

$$c(x) = (2\pi)^{-\frac{d}{2}} \int \exp\{ i(k,x) \} \left(k^2 \prod_{s=1}^{1} (k^2 + m_j^2) \right)^{-1} dk \quad .$$

If we prove with the help of a cluster expansion that the effective Lagrangian depends on **div** $\varphi(x)$ then the problem of the thermodynamic limit is solved in the case (ii).

REFERENCES

1. J.R.Fröhlich, Y.M.Park, *Commun.Math.Phys.*, 59, 235-266(1978).
2 D.Brydges, *Commun.Math.Phys.*, **58**, 311-350(1978).
3. J.Jmbrie, *Commun.Math.Phys.*, **87**, 515-565(1983).
4. D.Brydges, *Commun.Math.Phys.*, **73**, 197-246(1980).
5. T.Kennedy, *Commun.Math.Phys.*, **92**,,269-299(1983).
6. J.R.Fontaine, *Commun.Math.Phys.*, 103, 241-527(1986).
7. W.I.Skrypnik, *Teor.Mat.Fiz.*, 69, 128-141(1986)
8. D.Ruelle: Statistical Mechanics.Rigorous results. W.A.Benjamin inc.,New-York, Amsterdam, 1969.